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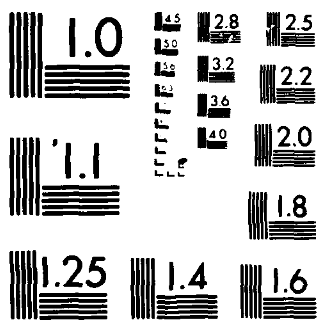

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STRUCTURE OF THE SET OF STEADY-STATE SOLUTIONS AND ASYMPTOTIC BEHAVIOUR OF SEMILINEAR HEAT EQUATIONS

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December 1981

(Received July 17, 1981)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

We give a precise geometrical description of the set of steady-state solutions for general classes of semilinear heat equations. This enables us to prove global results about the asymptotic behaviour of the solutions of the initial value problem.

AMS (MOS) Subject Classification: 35K60

Key Words: semilinear heat equations, semilinear elliptic equations, steady-state solutions, maximum principle.

Work Unit Number 1 - Applied Analysis

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Sponsored by the United States Army under Contract No. DAAG29-80-0041.

SIGNIFICANCE AND EXPLANATION

Semilinear heat equations - i.e. heat equations perturbed by a nonlinearity acting only on the lowest order term - arise in many contexts. It is well known that when studying the asymptotic behaviour of the solutions as the time t tends to infinity, a crucial role is played by the steady-state solutions. In this paper we present a global geometrical description of the set of steady-state solutions and this description enables us to give global results on the asymptotic behaviour of the solutions of the initial value problem.

STRUCTURE OF THE SET OF STEADY-STATE SOLUTIONS AND ASYMPTOTIC BEHAVIOUR OF SEMILINEAR HEAT EQUATIONS

P. L. Lions*

Introduction: Let Ω be a bounded, smooth, connected domain in $\mathbb{R}^N$. We consider both the initial value problem:

$$\text{(IVP)} \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(u) \quad \text{in} \quad \Omega \times (0,+\infty) \\ u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty),\ u(x,0) = u_0(x) \quad \text{in} \quad \bar{\Omega} \end{cases}$$

and the stationary problem:

$$\text{(SP)} \qquad -\Delta u = g(u) \quad \text{in} \quad \Omega \quad , \quad u = 0 \quad \text{on} \quad \partial\Omega$$

where u_0 is some given initial condition and f is a given nonlinearity of class C^1.

To explain our results, let us take, to simplify, the case when f is bounded on $\mathbb{R}$. We denote by S the set of solutions of (SP) (in $C^2(\bar{\Omega})$) and by S_+ (resp. S_-, resp. S_0) the set of functions u in S such that:

$$\lambda_1(-\Delta - f'(u)) > 0 \text{ (resp. } < 0, \text{ resp. } = 0) \quad ;$$

where $\lambda_1(-\Delta + c(x))$ denotes the first eigenvalue of $-\Delta + c(x)$ acting over $H_0^1(\Omega)$.

We prove here that we have:

i) S_+ consists of a at most countable number of isolated points,

ii) Every closed connected subset of S_0 is a totally ordered C^1 curve,

iii) If C is a connected subset of S_- then $\bar{C}$, the closure of C, is contained in S_-. And this implies that for every u_0, the ω-limit set $\omega(u_0)$ (i.e., the set of functions u such that there exists $t_n \underset{n}{\to} \infty$ with $u(\cdot,t_n) \underset{n}{\to} u(\cdot)$) is contained either in S_+, either in S_0, or in S_-.

We next define:

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We next define:

$I_+ = \{u_0 \ , \ \omega(u_0) \subset S_+\}$

$I_- = \{u_0 \ , \ \omega(u_0) \subset S_-\}$

$I_0 = \{u_0 \ , \ \omega(u_0) \subset S_0\}$.

We then prove that we have:

(i) $I_+ \cup I_0$ contains an open dense set;

(ii) On $I_+ \cup I_0$, $\omega(u_0)$ is a singleton;

(iii) If $u_0 \in I_-$, then there exists $\varepsilon > 0$, $v_+ \in S_+ \cup S_0$, $v_- \in S_+ \cup S_0$ such that, for all $\tilde{u}_0 \in B(u_0,\varepsilon) = \{\|u - u_0\| < \varepsilon\}$ (for some norm $\|\cdot\|$ precised below), we have:

$$\begin{cases} \tilde{u}_0 > u_0,\ \tilde{u}_0 \not\equiv u_0 \implies \omega(\tilde{u}_0) = \{v_+\} \\ \tilde{u}_0 < u_0,\ \tilde{u}_0 \not\equiv u_0 \implies \omega(\tilde{u}_0) = \{v_-\} \end{cases} .$$

In addition we can identify v_+ and v_-: for example, v_+ is the minimum element of S above any function of $\omega(u_0)$. In particular in view of (i), (ii), generically in u_0, the solution $u(x,t)$ of (IVP) converges as time goes to $+\infty$ to a stable solution (in a linearized sense) of (SP).

This study is, in some sense, the sequel of [13], where we studied the case when f is convex; and in the study of the (IVP) we will use strongly some results of [13].

The fact that, on I_0, $\omega(u_0)$ reduces to only one point is related to a recent work of J. K. Hale and P. Massatt [10] (and actually can be deduced from [10] and the description of S_0 given above). The only global results describing the instability of steady-state solutions in S_- are given in H. Fujita [9], P. L. Lions [13], and D. Henry [11], [12] - let us remark that in [9] only a very special case is studied, while in D. Henry [11], [12], it is assumed that not only S_0 is empty, but that, for all u in S_-, the linearized operator $-\Delta - f'(u)$ is one to one. Finally let us mention the related works of H. Matano [14], [15]; N. Chafee [4]; N. Chafee and E. Infante [5]; W. H. Fleming [8]; M. G. Crandall, P. Fife and L. A. Peletier [6]; M. Bertsch, P. L. Lions and L. A. Peletier [3].

In section I below, we study the sets S_+, S_-, S_0 and we prove claims i), ii) and iii). In section II, we give the proofs of the assertions concerning the asymptotic behaviour of the solutions of (IVP) in the case when f is such that the orbits are compact. Finally, in section III, we give various remarks and extensions.

Remark: The results and proofs given below remain trivially valid if the operator $-\Delta$ is replaced by a general self-adjoint operator A

$$A = -\sum_{i,j}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + c(x)$$

(with (a_{ij}) uniformly ellliptic); if $f(t)$ is replaced by a nonlinearity of the form $f(x,t)$ and if the Dirichlet boundary condition is replaced by a general boundary condition (insuring the maximum principle) as for example:

$$\frac{\partial u}{\partial n} + b(x)u = \varphi(x) \quad \text{on} \quad \partial\Omega$$

where n is the unit outward normal, $b \geq 0$, φ is given.

Let us also mention that the results of section I remain true for a general second-order elliptic operator (even if not self-adjoint).

I. Structure of the set of steady-state solutions

Let $f \in C^1(\mathbb{R})$, we denote by S the set of solutions $u \in C^2(\bar{\Omega})$ of (SP):

(SP) $\qquad -\Delta u = f(u)$ in Ω, $u \in C^2(\bar{\Omega})$, $u = 0$ on $\partial\Omega$.

If we denote by $\lambda_1(-\Delta + c(x))$ the first eigenvalue of the operator $(-\Delta + c)$ on the space $H_0^1(\Omega)$ (for any $c \in L^\infty(\Omega)$); then we introduce S_+ (resp. S_-, resp. S_0):

$S_+ = \{u \in S,\ \lambda_1(-\Delta - f'(u)) > 0\}$

$S_- = \{u \in S,\ \lambda_1(-\Delta - f'(u)) < 0\}$

$S_0 = \{u \in S,\ \lambda_1(-\Delta - f'(u)) = 0\}$.

In the result which follows, the topology is the one of the space $C_0(\bar{\Omega}) = \{u \in C(\bar{\Omega}),\ u = 0 \text{ on } \partial\Omega\}$.

Theorem I.1:

i) S_+ consists of a at most countable number of isolated points,

ii) <u>Every closed connected subset of</u> S_0 <u>is a totally ordered</u> C^1 <u>curve (for the partial order:</u> $u \leq v$, <u>if</u> $u(x) \leq v(x)\ \forall x \in \bar{\Omega}$),

iii) <u>If</u> C <u>is a connected subset of</u> S_- <u>then</u> $\bar{C} \subset S_-$,

iv) <u>For each</u> $u \in S_-$, <u>if the set</u> $M_+(u) = \{\tilde{u} \in S,\ \tilde{u} \geq u,\ \tilde{u} \not\equiv u\}$ <u>is not empty then it has a minimum element</u> $m_+(u)$ <u>and</u> $m_+(u) \in S_+ \cup S_0$. <u>Similarly, if</u> $M_-(u) = \{\tilde{u} \in S,\ \tilde{u} \leq u,\ \tilde{u} \not\equiv u\} \neq \emptyset$, <u>then this set has a maximum element</u> $m_-(u)$ <u>and</u> $m_-(u) \in S_+ \cup S_0$.

v) <u>If</u> C <u>is a connected component of</u> S_- <u>such that there exists</u> $u \in C$ <u>with</u> $M_+(u) \neq \emptyset$ (resp. $M_-(u) \neq \emptyset$) <u>then</u> $M_+(v) \neq \emptyset$ (resp. $M_+(v) \neq \emptyset$) <u>for all</u> $v \in C$ <u>and</u> m_+ (resp. m_-) <u>is constant on</u> C.

<u>Remark I.1</u>: By a simple use of the strong maximum principle (and of Hopf maximum principle) we have:

$$(1) \qquad m_+(u) > u > m_-(u) \ \text{ in } \Omega,\ \frac{\partial}{\partial n}(m_+(u)) < \frac{\partial u}{\partial n} < \frac{\partial}{\partial n}(m_-(u)) \ \text{ on } \partial\Omega \ .$$

Before giving the proof of Theorem I.1, we mention first an easy application of Theorem I.1 and state a result insuring that $M_+(u)$ or $M_-(u)$ are not empty.

<u>Corollary I.1</u>: <u>Let</u> C <u>be a closed connected subset of</u> S <u>then either</u> $C \subset S_+$, <u>either</u> $C \subset S_-$ <u>or</u> $C \subset S_0$.

<u>Proof</u>: If $C \cap S_+ \neq \emptyset$ then because of i) we have obviously:

$$C = \{u\} \subset S_+ \ .$$

On the other hand if $C \cap S_- \neq \emptyset$, then by the preceding argument we have necessarily: $C \cap S_+ = \emptyset$. Thus:

$$C = (C \cap S_-) \cup (C \cap S_0),\ (C \cap S_-) \cap (C \cap S_0) = \emptyset \ .$$

But in view of ii), $C \cap S_-$ is closed since $\overline{C \cap S_-} \subset S_-$. On the other hand, C and S_0 being closed, $C \cap S_0$ is closed. Now, since C is connected this implies: $C \cap S_-$ or $C \cap S_0$ is empty; and since we assumed $C \cap S_- \neq \emptyset$, we conclude: $C \cap S_0 = \emptyset$ or $C \subset S_-$.

<u>Proposition I.1</u>: <u>Let</u> $u \in S_-$.

(i) <u>Then</u> $M_+(u)$ <u>is not empty if and only if there exists</u> $w \in C^2(\bar{\Omega})$ <u>such that</u>:

$$(2) \qquad -\Delta w \geq f(w) \ \text{ in } \Omega,\ w \geq u \ \text{ in } \bar{\Omega},\ w \not\equiv u \ .$$

(ii) Similarly $M_-(u)$ is not empty if and only if there exists $w \in C^2(\bar{\Omega})$ such that

(3) $$-\Delta w \leq f(w) \text{ in } \Omega, \; w \leq u \text{ in } \bar{\Omega}, \; w \not\equiv u \; .$$

The proof of Proposition I.1 will be given below, together with the proof of iv).

Remark I.2: If $f(t)$ satisfies:

(4) $$\overline{\lim_{t\to+\infty}} f(t)t^{-1} < \lambda_1 = \lambda_1(-\Delta)$$

then $M_+(u)$ is not empty for all $u \in S_-$. Indeed let μ be such that:

$$\overline{\lim_{t\to+\infty}} f(t)t^{-1} < \mu < \lambda_1 \; ;$$

we have for some $C > 0$:

$$f(t) \leq \mu t + C \;, \quad \text{for all } t \geq 0 \; .$$

Next, let w_λ be the solution of

$$-\Delta w_\lambda - \mu w_\lambda = \lambda C \text{ in } \Omega, \; w_\lambda \in C^2(\bar{\Omega}), \; w_\lambda = 0 \text{ on } \partial\Omega$$

with $\lambda \geq 1$.

For λ large, we have: $w_\lambda > u$ in Ω; and since we have

$$-\Delta w_\lambda = \mu w_\lambda + C\lambda \geq \mu w_\lambda + C \geq f(w_\lambda) \text{ in } \Omega \;,$$

we conclude by a simple application of Proposition I.1. Similarly, if $f(t)$ satisfies:

(5) $$\overline{\lim_{t\to-\infty}} f(t)t^{-1} < \lambda_1 \; (= \lambda_1(-\Delta)) \;,$$

then $M_-(u)$ is not empty for all $u \in S_-$.

We now turn to the proof of Theorem I.1:

Proof of i): Since, by definition, for every $u \in S_+$ the linearized operator $(-\Delta - f'(u))$ is one to one, u is an isolated point in S. In addition, in view of the equation (SP) and the Schauder estimates the set $\{u \in S, \|u\|_{C(\bar{\Omega})} \leq R\}$ is compact for every $R < +\infty$. These two facts prove i).

Proof of ii): We first prove that if C is a closed connected subset of S_0 then C is totally ordered. Of course, we may assume that C contains more than only one point; in this case each point u of C is an accumulation point, that is:

$$\forall u \in C \quad \exists (u_n)_n \in C, \; u_n \not\equiv u, \; u_n \xrightarrow[n]{} u \; .$$

Then we have

$$\begin{cases} -\Delta(u_n-u) = \{(f(u_n) - f(u))(u_n-u)^{-1}\}(u_n-u) \quad \text{in } \Omega \\ u_n-u \in C^2(\bar{\Omega}),\ u_n-u = 0 \ \text{on } \partial\Omega,\ u_n-u \not\equiv 0 \ ; \end{cases}$$

where $(f(u_n) - f(u))(u_n - u)^{-1} = c_n(x)$ ($= f'(u(x))$ if $u_n(x) = u(x)$). Thus 0 is some eigenvalue of the operators $(-\Delta - c_n)$ and since c_n converges (uniformly) to $c(x) = f'(u(x))$, we deduce:

$$\lambda_1(-\Delta - c_n) = 0 \ \text{for } n \text{ large enough} \ ;$$

but this implies $\pm(u_n-u) > 0$ in $\bar{\Omega}$. Thus, we have shown that:

$$\forall\, u \in C,\ \exists \varepsilon > 0,\ \forall\, v \in C\ \|v-u\| < \varepsilon \Longrightarrow \lambda_1\left(-\Delta - \frac{f(v)-f(u)}{v-u}\right) = 0 \ .$$

Next, we define the map from $C \times C$ into $\mathbb{R}$:

$$\Lambda(u,v) = \lambda_1\left(-\Delta - \frac{f(v)-f(u)}{v-u}\right) \ .$$

It is clear that Λ is continuous, therefore the set $\tilde{C} = \Lambda^{-1}\{0\}$ is closed. But on the other hand one can prove in the same way as above that if $\Lambda(u,v) = 0$ then: $\exists\, \varepsilon > 0$ $\forall\, \tilde{u}, \tilde{v} \in C$ $\|\tilde{u}-u\| + \|\tilde{v}-v\| < \varepsilon \Longrightarrow \Lambda(\tilde{u},\tilde{v}) = 0$. (Indeed 0 is an eigenvalue of $\{-\Delta - (f(\tilde{v})-f(\tilde{u}))(\tilde{v}-\tilde{u})^{-1}\}$ and as ε goes to 0, $(f(\tilde{v}) - f(\tilde{u}))(\tilde{v}-\tilde{u})^{-1}$ converges to $(f(v)-f(u))(v-u)^{-1}$; since $\Lambda(u,v) = 0$, this implies $\Lambda(\tilde{u},\tilde{v}) = 0$ for ε small enough.) This shows that $\Lambda^{-1}\{0\} = \tilde{C}$ is also open (for the relative topology on $C \times C$) but since $C \times C$ is connected and since $\Lambda(u,u) = \lambda_1(-\Delta - f'(u)) = 0$ ($C \subset S_0$) we deduce: $\tilde{C} = C \times C$, or in other words:

$$\lambda_1(-\Delta - (f(u) - f(v))(u-v)^{-1}) = 0,\ \forall u,\ v \in C \ .$$

Since we have for all $u, v \in C$

$$\begin{cases} -\Delta(u-v) = \{(f(u) - f(v))(u-v)^{-1}\}(u-v) \quad \text{in } \Omega \\ u-v \in C^2(\bar{\Omega}),\ u-v = 0 \ \text{on } \partial\Omega \ , \end{cases}$$

we proved that if $u, v \in C$, $u \neq v$ then necessarily we have

either: $\quad u > v$ in Ω, $\frac{\partial u}{\partial n} < \frac{\partial v}{\partial n}$ on $\partial\Omega$

or: $\quad u < v$ in Ω, $\frac{\partial u}{\partial n} > \frac{\partial v}{\partial n}$ on $\partial\Omega$;

that is C is totally ordered.

We now prove that C is a C^1 curve: without loss of generality we may assume that C contains more than one point. Then we introduce for all $v \in C$:

$$t(v) = \int_\Omega v(x)dx \quad .$$

This defines a continuous map from C into $\mathbb{R}$: its range is some interval (t_0,t_1) (if C has a maximum element, we take $(t_0, t_1]$ and if C has a minimum element, we take $[t_0,t_1)$). Since C is totally ordered, it is clear that:

$$t(v) = t(v') \Longrightarrow v = v' \quad .$$

Thus the map from C into I $(v \mapsto t(v))$ is continuous and one to one. In addition since C is totally ordered, we have: $t(v) > t(v') \Longrightarrow v > v'$ in Ω. We may now define a parametrization of C: $I \ni t \mapsto v_t$ where v_t is given by the solution of $t(v_t) = t$. It is very easy to check that v_t is continuous for $t \in I$.

We now prove that the map $(t \overset{V}{\mapsto} v_t)$ is C^1 on I and that $V'(t) = w_t$ where w_t is the normalized first eigenfunction of:

$$(6) \qquad \begin{cases} -\Delta w_t = f'(v_t)w_t \quad \text{in} \quad \Omega, \; w_t \in C^2(\bar{\Omega}) \\ w_t > 0 \quad \text{in} \quad \Omega, \; w_t = 0 \quad \text{on} \quad \partial\Omega, \; \int_\Omega w_t(x)dx = 1 \quad . \end{cases}$$

since the continuity of w_t with respect to t is a standard consequence of the continuity of $f'(v_t)$, we will only prove that: $\frac{1}{h}(V(t+h) - V(t)) \xrightarrow[h\to 0_+]{} w_t$.

But we have:

$$(7) \qquad \begin{cases} -\Delta(\frac{1}{h}(v_{t+h} - v_t)) = \{(f(v_{t+h}) - f(v_t))(v_{t+h} - v_t)^{-1}\}\{\frac{1}{h}(v_{t+h} - v_t)\} \\ \frac{1}{h}(v_{t+h} - v_t) \in C^2(\bar{\Omega}), \; \frac{1}{h}(v_{t+h} - v_t) = 0 \quad \text{on} \quad \partial\Omega, \; \frac{1}{h}(v_{t+h} - v_t) > 0 \quad \text{in} \quad \Omega \quad ; \end{cases}$$

and

$$(8) \qquad \int_\Omega \frac{1}{h}(v_{t+h} - v_t)(x)dx = 1 \quad .$$

If we denote by: $w_t^h = \frac{1}{h}(v_{t+h} - v_t)$. In view of (7) and (8), we have:

$$\|w_t^h\|_{L^1(\Omega)} \leq C, \; \|-\Delta w_t^h\|_{L^1(\Omega)} \leq C \quad (\text{for some } C \text{ ind. of } h) \quad .$$

From well-known regularity results, this implies:

$$\|w_t^h\|_{L^p(\Omega)} \leq C, \quad \text{for all} \quad 1 \leq p < \frac{N}{N-2} \; (<\infty \;\text{ if }\; N = 1,2) \quad ;$$

but this implies, using (7): $\|-\Delta w_t^h\|_{L^p(\Omega)} \leq C$, for all $1 \leq p < \frac{N}{N-2}$ and by a straightforward bootstrap argument and by Schauder estimates we obtain:

$$\|w_t^h\|_{C^{2,\alpha}(\bar{\Omega})} \leq C \ (\forall\ 0 < \alpha < 1) \ .$$

Thus taking if necessary a subsequence, $w_t^h \underset{C^2(\bar{\Omega})}{\to} w_t$ solution of (6),

Remark I.3: It is easy to deduce from the above proof that the curve C has the same regularity than f (if $f \in C^k$, then C is of class C^k, for all $1 \leq k \leq \infty$).

Proof of iii): Let C be a connected subset of S_-, we may assume without loss of generality that C contains more than one point. Let (u_n) be a converging sequence in C, $u_n \underset{n}{\to} u$. It is clear that $u \in S_- \cup S_0$. Suppose that $u \in S_0$ and let us try to obtain a contradiction.

Each u_n is an accumulation point in C, thus there exists $u_n^m \in C$ such that: $u_n^m \underset{m}{\to} u_n$, $u_n^m \not\equiv u_n$.

$$\begin{cases} -\Delta(u_n^m - u_n) = \{(f(u_n^m) - f(u_n))(u_n^m - u_n)^{-1}\}(u_n^m - u_n) & \text{in } \Omega \\ u_n^m - u_n \in C^2(\bar{\Omega}),\ u_n^m - u_n \not\equiv 0,\ u_n^m - u_n = 0 & \text{on } \partial\Omega \ . \end{cases}$$

Therefore 0 is some eigenvalue of the operator $(-\Delta - c_n^m)$ where $c_n^m = \{(f(u_n^m) - f(u_n))(u_n^m - u_n)^{-1}\}$. Since c_n^m converges, as m goes to infinity, to $c_n = f'(u_n)$; we deduce that 0 is an eigenvalue of $(-\Delta - f'(u_n))$. Now, since $u_n \underset{n}{\to} u$ and thus $f'(u_n) \underset{n}{\to} f'(u)$ and since $\lambda_1(-\Delta - f'(u)) = 0$, this would imply that for n large: $\lambda_1(-\Delta - f'(u_n)) = 0$; and this contradicts the assumption: $u_n \in C \subset S_-$. The contradiction proves iii).

Remark I.4: We proved in fact that if $u_n \in S_-$ and u_n is some accumulation point in S, then all limit points of the sequence $(u_n)_n$ lie in S_-.

Proof of iv) and of Proposition I.1: We will only prove the assertions concerning m_+, M_+. We first remark that if $M_+(u)$ is not empty then any $\tilde{u}$ in $M_+(u)$ satisfies (2). Thus it remains to prove that if there exists w satisfying (2), then there exists a minimum element in S above u. We will prove that this minimum element $m_+(u)$ belongs to $S_+ \cup S_0$.

To prove the existence of a minimum solution in S above u, we will adapt some general results of H. Amann [1], [2]. Since $u \in S_-$, there exists $\lambda_1 < 0$ and $v_1 \in C^2(\bar{\Omega})$ such that:

$$\begin{cases} -\Delta v_1 = f'(u)v_1 + \lambda_1 v_1 \quad \text{in } \Omega,\ v_1 \in C^2(\bar{\Omega}) \quad , \\ v_1 > 0 \quad \text{in } \Omega,\ v_1 = 0 \quad \text{on } \partial\Omega \quad . \end{cases}$$

Thus for ε small enough $(0 < \varepsilon \leq \varepsilon_1)$:

$$-\Delta(u+\varepsilon v_1) = f(u) + f'(u)\varepsilon v_1 + \varepsilon\, \lambda_1 v_1 \quad \text{in } \Omega$$

$$\leq f(u+\varepsilon v_1) \quad \text{in } \Omega \quad .$$

On the other hand, if w satisfies (2), from the strong maximum principle we deduce:

$$w(x) > u(x) \quad \text{in } \Omega,\ \frac{\partial w}{\partial n} < \frac{\partial u}{\partial n} \quad \text{on } \partial\Omega \quad .$$

Therefore for ε small enough $(0 < \varepsilon \leq \varepsilon_2)$:

$$u + \varepsilon v_1 < w \quad \text{in } \Omega,\ \frac{\partial w}{\partial n} < \frac{\partial}{\partial n}(u + \varepsilon v_1) \quad \text{on } \partial\Omega \quad .$$

Hence we obtained, for $0 < \varepsilon \leq \varepsilon_0 = \min(\varepsilon_1,\varepsilon_2)$,

$$-\Delta(u + \varepsilon v_1) \leq f(u + \varepsilon v_1) \quad \text{in } M, \quad u + \varepsilon v_1 < w \quad \text{in } \Omega \quad .$$

We will denote by K a positive constant such that:

$$f'(t) + Kt \geq 0, \quad \text{for } t \in [-\|u\|_{C(\bar{\Omega})}, +\|w\|_{C(\bar{\Omega})}] \quad .$$

And we introduce the standard iterative method: $(0 < \varepsilon \leq \varepsilon_0)$

$$\begin{cases} u_0^\varepsilon = u + \varepsilon v_1 \quad ; \\ -\Delta u_{n+1}^\varepsilon + K u_{n+1}^\varepsilon = f(u_n^\varepsilon) + K u_n^\varepsilon \quad \text{in } \Omega \quad , \\ u_{n+1}^\varepsilon \in C^2(\bar{\Omega}),\ u_{n+1}^\varepsilon = 0 \quad \text{on } \partial\Omega \quad . \end{cases}$$

It is then obvious to show that:

$$\begin{cases} u < u_0^\varepsilon < u_1^\varepsilon < \cdots < u_n^\varepsilon < u_{n+1}^\varepsilon < \cdots < w \quad \text{in } \Omega \\ u_n^\varepsilon \to u^\varepsilon \in S \quad \text{in } C^2(\bar{\Omega}) \quad , \end{cases}$$

and in addition: $u_n^\varepsilon \leq u_n^{\varepsilon'}$ $\forall\, n \geq 1$, $\forall\, 0 < \varepsilon \leq \varepsilon' \leq \varepsilon_0$ and thus: $u^\varepsilon \leq u^{\varepsilon'}$. Hence, as ε goes to 0, u^ε converges in $C^2(\bar{\Omega})$ to $\tilde{u} \in S$.

We now prove that for all $\varepsilon > 0$ $u^\varepsilon \in S_+ \cup S_0$, thus $\tilde{u} \in S_+ \cup S_0$ and we will deduce that $u^\varepsilon \equiv \tilde{u}$ for ε small enough.

Indeed we have:

$$-\Delta(u^\varepsilon - u^\varepsilon_{n+1}) + K(u^\varepsilon - u^\varepsilon_{n+1}) = f(u^\varepsilon) + Ku^\varepsilon - f(u^\varepsilon_n) - Ku^\varepsilon_n$$

$$\geq f(u^\varepsilon) + Ku^\varepsilon - f(u^\varepsilon_{n+1}) - Ku^\varepsilon_{n+1}$$

$$\geq \{(f(u^\varepsilon) - f(u^\varepsilon_{n+1}))(u^\varepsilon - u^\varepsilon_{n+1})^{-1} + K\}(u^\varepsilon - u^\varepsilon_{n+1}) \quad \text{in } \Omega$$

and $u^\varepsilon - u^\varepsilon_{n+1} > 0$ in M, $u^\varepsilon - u^\varepsilon_{n+1} \in C^2(\bar{\Omega})$, $u^\varepsilon - u^\varepsilon_{n+1} = 0$ on $\partial\Omega$.

This implies: $\lambda_1(-\Delta - c^\varepsilon_{n+1}) \geq 0$,

where $c^\varepsilon_{n+1} = \{(f(u^\varepsilon) - f(u^\varepsilon_{n+1}))(u^\varepsilon - u^\varepsilon_{n+1})^{-1}\}$.

Next, as n goes to ∞, $c^\varepsilon_{n+1} \underset{n}{\to} f'(u^\varepsilon)$ and thus $u^\varepsilon \in S_+ \cup S_0$. Hence $\tilde{u} \in S_+ \cup S_0$, $\tilde{u} \not\equiv u$, $u \leq \tilde{u} \leq u^\varepsilon$ in $\bar{\Omega}$.

But the maximum principle shows that:

$$u(x) < \tilde{u}(x) \text{ in } \Omega, \ \frac{\partial u}{\partial n} > \frac{\partial \tilde{u}}{\partial n} \text{ on } \partial\Omega \ ;$$

and therefore we have for ε small enough: $u + \varepsilon v_1 \leq \tilde{u}$ in Ω, this yields: $u^\varepsilon_n \leq \tilde{u}$, $u^\varepsilon \leq \tilde{u}$ and $u^\varepsilon \equiv \tilde{u}$ for ε small enough.

Now for any $\bar{u} \in S$ such that $\bar{u} \geq u$, $\bar{u} \not\equiv u$; we obtain from the strong and Hopf maximum principle:

$$\bar{u} > u \text{ in } \Omega, \ \frac{\partial \bar{u}}{\partial n} < \frac{\partial u}{\partial n} \text{ on } \partial\Omega$$

and therefore for ε small enough: $u + \varepsilon v_1 \leq \bar{u}$ in Ω. This implies: $u^\varepsilon_n \leq \bar{u}$, $\forall n$ and since $u^\varepsilon_n \underset{n}{\to} \tilde{u}$ we deduce $\tilde{u} \leq \bar{u}$. This shows that $\tilde{u}$ is the minimum element of S above u. We will denote it by $m_+(u)$; and we already showed that $m_+(u) \in S_+ \cup S_0$.

<u>Remark I.5</u>: Other arguments for the existence of $m_+(u)$ can be given but we prefer the above one since it yields a constructive existence proof.

<u>Proof of v)</u>: Again we will only prove the assertions concerning m_+, M_+. Let C be a connected component of S_- and suppose there exists u_0 with $M_+(u_0) \neq \emptyset$. Let C' be the connected component of the set $\{\tilde{u} \in S_-, M_+(\tilde{u}) \neq \emptyset\}$ containing u_0. We first show that C' is open (for the relative topology): indeed if $u \in C'$, $m_+(u)$ satisfies:

$$m_+(u) > u \text{ in } \Omega,\ \frac{\partial u}{\partial n} > \frac{\partial}{\partial n}(m_+(u)) \text{ on } \partial\Omega \ .$$

Therefore for $v \in S$, v near u; we still have: $v < m_+(u)$ in $\bar{\Omega}$, thus $M_+(v) \neq \emptyset$ and this shows that C' is open.

We next show that m_+ is continuous on C': let $u_n \in C'$, $u_n \underset{n}{\to} u$ and $u \in C'$; let us prove that $m_+(u_n) \underset{n}{\to} m_+(u)$. We first remark that for n large enough, as we proved above, we have: $u_n < m_+(u)$ in $\bar{\Omega}$. Thus: $u_n < m_+(u_n) < m_+(u)$ in $\bar{\Omega}$, for n large enough. This proves in particular that $m_+(u_n)$ is bounded in $L^\infty(\Omega)$, and using the equation (SP) and regularity estimates we deduce that $m_+(u_n)$ is bounded in $C^{2,\alpha}(\bar{\Omega})$ ($\forall\ 0 < \alpha < 1$). Now (taking if necessary a subsequence) $m_+(u_n)$ converges in $C^2(\bar{\Omega})$ to $\tilde{u} \in S_+ \cup S_0$ and $u < \tilde{u} < m_+(u)$.

Since $\tilde{u} = S_+ \cup S_0$, $\tilde{u} \ngtr u$ and thus from the definition of $m_+(u)$ this shows that $\tilde{u} = m_+(u)$. This proves the continuity of m_+ on C'.

We now prove that m_+ is constant on C' and thus

$$m_+(u) = m_+(u_0),\ \forall\, u \in C' \ .$$

Indeed, $m_+(C')$ is a connected set $\subset S_+ \cup S_0$. But because of i), if $m_+(C') \cap S_+ \neq \emptyset$ then $m_+(C') = \{m_+(u_0)\} \subset S_+$. And if $m_+(C') \cap S_+ = \emptyset$, we then have $m_+(C') \subset S_0$; and from ii), we deduce that $m_+(C')$ is totally ordered. This will enable us to show that m_+ is locally constant and this concludes the proof since m_+ is continuous and C' is connected. Let $v \in C'$; if m_+ is not constant in a relative neighborhood of v, then there exists $v_n \in C'$, $v_n \underset{n}{\to} v$ and $m_+(v_n) \neq m_+(v)$. Since $m_+(v_n)$ and $m_+(v)$ can be compared we have:

either $\qquad m_+(v_n) < m_+(v)$ in Ω

or $\qquad m_+(v_n) > m_+(v)$ in Ω .

If the first case happens for n large enough, recalling that $m_+(v_n) \underset{n}{\to} m_+(v)$, we should have: $v < m_+(v_n)$ in Ω. And this contradicts the definition of $m_+(v)$.

Now, if the second case happens for n large enough, recalling that $v_n \underset{n}{\to} v < m_+(v)$ in Ω (and $\frac{\partial v}{\partial n} > \frac{\partial}{\partial n}(m_+(v))$ on $\partial\Omega$); this would imply: $v_n < m_+(v) < m_+(v_n)$ in Ω. And this contradicts the definition of $m_+(v_n)$. Thus m_+ is locally constant in C' and this shows that m_+ is constant on C': $m_+(u) = m_+(u_0)$, $\forall\, u \in C'$.

We may now conclude by proving that C' is closed (for the relative topology) and since C is connected, this will show that $C = C'$. Therefore let $u_n \in C'$, $u_n \underset{n}{\to} u \in C$. We have just proved

$$u_n < m_+(u_n) = m_+(u_0) \in S_+ \cup S_0 \quad .$$

Thus: $u < m_+(u_0) \in S_+ \cup S_0$, and $u \not\equiv m_+(u_0)$. This shows that $M_+(u) \neq \emptyset$ and that $u \in C'$.

Remark I.6: We would like to point out that because of the local compactness of S, Theorem I.1 is still valid (with the same proof) for the topology of any space like $W_0^{1,\infty}(\Omega)$, $C_0^1(\bar{\Omega})$, $C_0^2(\bar{\Omega})$ - where we denote by X_0 the subspace of any functional space X of functions vanishing on $\partial\Omega$-.

Remark I.7: As it was mentioned in the Introduction, Theorem I.1 and its proof are still valid for any uniformly elliptic second-order operaotr instead of $-\Delta$, for general nonlinearities $f(x,t)$ instead of $f(x)$ and for general boundary conditions (satisfying the maximum principle).

II. Asymptotic behaviour for quasi-bounded nonlinearities.

We consider now the (IVP)

$$\text{(IVP)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(u) \quad \text{in} \quad \Omega \times (0,+\infty), \; u \in C^2(Q) \cap C(\bar{Q}) \\ u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty), \; u(x,0) = u_0(x) \quad \text{in} \quad \bar{\Omega} \end{cases}$$

where $Q = \Omega \times (0,+\infty)$, u_0 is some given initial condition in the space $X = W_0^{1,\infty}(\Omega)$ (for example).

We will assume that $f(\in C^1(\mathbb{R}))$ satisfies:

$$\text{(9)} \qquad \overline{\lim_{|t|\to\infty}} \; f(t)t^{-1} < \lambda_1 = \lambda_1(-\Delta) \quad .$$

This insures for example that, for any u_0 in X, there exists a unique solution $u(x,t)$ of (IVP) and that:

$$\|u(x,t)\|_{C(\bar{\Omega})} \leq C \quad \text{(indep. of } t > 0) \quad .$$

This implies that $u(\cdot,t)$ is bounded in $C^{2,\alpha}(\overline{\Omega})$ (for $t > \delta > 0$) for all $0 < \alpha < 1$. In particular the orbit $(u(\cdot,t))_{t>\delta}$ is compact in X (for all $\delta > 0$).

Of course $u(x,t)$ defines a (nonlinear) semigroup: $u(\cdot,t) = S(t)u_0$. Finally, since the orbit is compact in X, by well-known results (see for example C. M. Dafermos [7]) we have, denoting by $\omega(u_0)$ the ω-limit set of $(S(t)u_0)_{t>0}$ i.e.,

$\omega(u_0) = \{u \in X, \exists\, t_n \underset{n}{\to} \infty\ u(\cdot,t_n) \underset{n}{\to} u\}$

(10) $$\omega(u_0) \text{ is a connected compact subset of } X .$$

In addition, since we have a Lyapunov function namely:

(11) $$S(v) = \int_\Omega \frac{1}{2}|Dv|^2 - F(v)dx \quad ,$$

where $F(t) = \int_0^t f(s)ds$; we have:

(12) $$\omega(u_0) \subset S; \ \frac{\partial u}{\partial t} \underset{t\to+\infty}{\to} 0 \ \text{ in } C^{1,\alpha}(\overline{\Omega}) \ (0 < \alpha < 1) \quad .$$

Now applying Corollary I.1, we see that only three *possibilities may happen:* i) $\omega(u_0) \subset S_+$ or ii) $\omega(u_0) \subset S_0$ or iii) $\omega(u_0) \subset S_-$. It is then natural to introduce the three sets (which are disjoint):

$$I_+ = \{u_0 \in X, \ \omega(u_0) \subset S_+\}$$
$$I_- = \{u_0 \in X, \ \omega(u_0) \subset S_-\}$$
$$I_0 = \{u_0 \in X, \ \omega(u_0) \subset S_0\} \quad ;$$

we just explained why we have: $X = I_+ \cup I_0 \cup I_-$.

We will denote by $B(u_0,\varepsilon) = \{u \in X, \|u-u_0\|_X < \varepsilon\}$. Our main result concerning the asymptotic behaviour of (IVP) is the following:

Theorem II.1: Under assumption (9) and if $f \in C^2(\mathbb{R})$ we have:

i) $I_+ \cup I_0$ contains an open dense set;

ii) For all u_0 in $I_+ \cup I_0$, $\omega(u_0)$ is a singleton;

iii) If $u_0 \in I_-$, there exists $\varepsilon > 0$ such that:

(13) $$\forall v \in B(u,\varepsilon),\ v > u_0,\ v \not\equiv u_0 \Longrightarrow \omega(v) = \{u^+\} \subset S_+ \cup S_0$$

where $u^+ = m^+(v)$, $\forall w \in \omega(u_0)$;

(14) $$\forall v \in B(u_0,\varepsilon),\ v < u_0,\ v \not\equiv u_0 \Longrightarrow \omega(v) = \{u^-\} \subset S_+ \cup S_0$$

where $u^- = m^-(w)$, $\forall w \in \omega(u_0)$.

In particular if $v \in B(u_0,\varepsilon)$, $v > u_0$ on $v < u_0$, $v \not\equiv u_0$ then $v \in I_+ \cup I_0$.

Remark II.1: Remark first that in view of Remark I.2 and assumption (9), m^+ and m^- are defined on S_- and are constant on each connected component of S_-; and thus m_+, m_- are constant on $\omega(u_0)$ since $\omega(u_0)$ is connected (for $u \in I_-$).

Remark II.2: The fact that $\omega(u_0)$ is a singleton, for u_0 in I_0 can be deduced from the fact that by part ii) of Theorem I.1 $\omega(u_0)$ is a C^1-curve and from general results on dynamical systems of J. Hale and P. Massalt [10]. However we give a different proof which is a trivial consequence of the fact that $\omega(u_0)$ is totally ordered.

Remark II.3: The above result shows in particular that, generically in X, $S(t)u_0 \underset{t\to\infty}{\to} u \in S_+ \cup S_0$ (that is a solution of (SP) such that: $\lambda_1(-\Delta - f'(u)) \geq 0$).

Remark II.4: The above result is still valid if we replace $W_0^{1,\infty}(\Omega)$ by $C_0^1(\bar{\Omega})$ or $C_0^2(\bar{\Omega})$ or the subspace E of $C_0(\bar{\Omega})$ defined by: $E = \bigcup_{\lambda>0} \lambda[-\delta,+\delta]$ where $\delta(x) = \mathrm{dist}(x,\partial\Omega)$ and where $[u,v] = \{w \in C_0(\bar{\Omega}),\ u \leq w \leq v\}$, E is equipped with the order unit norm: $\|u\| = \inf(\lambda > 0 \ / \ -\lambda\delta \leq u \leq \lambda\delta)$ (E with this norm is a Banach space - for more details, see H. Amann [2]).

Remark II.5: Similar results hold for general self adjoint uniformly elliptic second-order operators, for general nonlinearities of the form $f(x,t)$ and for general boundary conditions preserving the maximum principle.

Before giving the proof of Theorem II.1, we mention the following standard comparison principle: let $u_0 \leq v_0$, $u_0 \not\equiv v_0$ then for all $t > 0$ we have:

$$S(t)u_0(x) < S(t)v_0(x) \text{ in } \Omega,\ \frac{\partial}{\partial n}(S(t)u_0) > \frac{\partial}{\partial n}(S(t)v_0) \text{ on } \partial\Omega .$$

Proof of ii): Because of (10) and of part i) of Theorem I.1, then it is trivial that $\omega(u_0)$ is a singleton if $u_0 \in I_+$. Next, let $u_0 \in I_0$, $\omega(u_0)$ is a compact connected set $\subset S_0$, thus $\omega(u_0)$ is totally ordered. If $\omega(u_0)$ is not a singleton, there exist u_1, u_2, u_3 in $\omega(u_0)$ such that: $u_1 < u_2 < u_3$ in Ω, $\frac{\partial}{\partial n}(u_1) > \frac{\partial}{\partial n}(u_2) > \frac{\partial}{\partial n}(u_3)$ on $\partial\Omega$. Since $u_1 \in \omega(u_0)$, there exists $(t_n)_n$ such that $u(x,t_n) \underset{n}{\to} u_1(x)$ in $C^1(\bar{\Omega})$. Therefore for n large enough, we have: $u(x,t_n) \leq u_2(x)$ in $\bar{\Omega}$. And this gives: $u(x,t) \leq S(t-t_n)u_2(x) = u_2(x)$, for $t \geq t_n$. And this contradicts the fact that $u_3 \in \omega(u_0)$.

Proof of iii): We will only prove the part concerning (13). Let $u_0 \in I_0$, we recall a few results proved in P. L. Lions [13]: for t large enough we have:

(15) $$\exists\alpha > 0,\ \lambda_1(-\Delta - f'(u(\cdot,t))) \leq -\alpha < 0$$

(this is true for "$t = +\infty$" since $\omega(u_0) \subset S_-$, and by continuity this remains true for t large).

Let $v_1(x,t)$ be a normalized eigenfunction of

$$\begin{cases} -\Delta v_1 = f'(u(x,t))v_1 + \lambda_1(t)v_1 \quad \text{in } \bar{\Omega} , \\ v_1 \in C^2(\bar{\Omega}),\ v_1 > 0 \ \text{in } \Omega,\ v_1 = 0 \ \text{on } \partial\Omega,\ |v_1|_{L^2(\Omega)} = 1 . \end{cases}$$

(and thus by (15), $\lambda_1(t) \leq -\alpha < 0$).

In [13], it is proved that: $\frac{\partial}{\partial t}(\lambda_1(t)) \xrightarrow[t\to+\infty]{} 0$, $\frac{\partial}{\partial t}(v_1(x,t)) \xrightarrow[t\to+\infty]{} 0$ in $C^1(\bar{\Omega})$. Then we have for $\varepsilon > 0$:

$$\frac{\partial}{\partial t}(u+\varepsilon v_1) - \Delta(u+\varepsilon v_1) = f(u) + f'(u)\varepsilon v_1 + \lambda_1 \varepsilon v_1 + \varepsilon \left(\frac{\partial}{\partial t} v_1\right)$$

and for ε small enough, we deduce:

$$\frac{\partial}{\partial t}(u+\varepsilon v_1) - \Delta(u+\varepsilon v_1) \leq f(u+\varepsilon v_1) - \frac{\alpha}{2}\varepsilon v_1 + \varepsilon \frac{\partial}{\partial t} v_1 .$$

Since for t large enough: $\frac{\partial}{\partial t}(v_1) \leq \frac{\alpha}{2} v_1$ in $\bar{\Omega}$; we obtain finally:

(16) $$\begin{cases} \exists\varepsilon_0 > 0,\ \exists T_0 > 0 \ \text{such that for } \varepsilon \in]0,\varepsilon_0],\ \text{for } t \geq T_0 \ \text{we have:} \\ \frac{\partial}{\partial t}(u+\varepsilon v_1) - \Delta(u+\varepsilon v_1) \leq f(u+\varepsilon v_1) \quad \text{in } \Omega . \end{cases}$$

Remark that we also have:

(17) $$\exists C \geq \gamma > 0 \ \text{such that:}\ \gamma\delta(x) \leq v_1(x,t) \leq C\delta(x) \ \text{in } \bar{\Omega}$$

(recall that $\delta(x) = \operatorname{dist}(x,\partial\Omega)$).

Next, let $u^+ = m^+(w)$ ($\forall w \in \omega(u_0)$, we already showed that m_+ is constant on $\omega(u_0)$ - see Remark II.1 above). A simple continuity argument shows that there exists $T > 0$, $\nu > 0$ such that:

$$u(x,T) + \nu\delta(x) \leq u^+(x) \quad \text{in } \bar{\Omega} .$$

Next let ε be such that, for $v \in B(u_0,\varepsilon)$, $S(T)v \leq u^+$ in $\bar{\Omega}$ (use the continuity of the map $S(T)$ from X into $C^1(\bar{\Omega})$).

We now take $v \in B(u_0,\varepsilon)$, $v \geq u_0$, $v \not\equiv u_0$ and we are going to prove that $S(t)v \underset{t\to\infty}{\to} u^+$. From the choice of ε above we surely h- $w \leq u^+$, $w \in S$ $\forall w \in \omega(v)$. From the definition of u^+, it just remains to prov if $w \in \omega(v)$ then there exists $\tilde{u} \in \omega(u_0)$ such that: $w \geq \tilde{u}$, $w \not\equiv \tilde{u}$. And this will be achieved with the help of (16) - (17). Indeed, we have:

$$S(T_0)v(x) > S(T_0)u_0(x) \quad \text{in} \quad \Omega, \ \frac{\partial}{\partial n}(S(T_0)v) < \frac{\partial}{\partial n}(S(T_0)u_0) \quad \text{on} \quad \partial\Omega \ ,$$

and thus there exists ε small enough such that:

$$S(T_0)v \geq S(T_0)u_0 + \varepsilon v_1(x,T_0) \quad \text{in} \quad \bar{\Omega} \ .$$

Next because of (16), this yields:

$$S(t)v(x) \geq u(x,t) + \varepsilon v_1(x,t) \quad \text{in} \quad \bar{\Omega}, \quad \text{for} \quad t \geq T_0 \ ;$$

or in view of (17):

$$S(t)v(x) \geq u(x,t) + \varepsilon\gamma\delta(x) \quad \text{in} \quad \bar{\Omega}, \quad \text{for} \quad t \geq T_0 \ .$$

Now it is easy to conclude, since if $S(t_n)v \underset{n}{\to} w \in \omega(v)$, there exists a subsequence $t_{n'}$ such that $u(\cdot,t_{n'}) \underset{n'}{\to} \tilde{u} \in \omega(u_0)$ and we have:

$$w \geq \tilde{u} + \varepsilon\gamma\delta \quad \text{in} \quad \bar{\Omega} \ ;$$

thus $w \not\equiv \tilde{u}$ and since we already know $w \leq u^+ = m^+(\tilde{u})$, we conclude from the definition of m^+: $w = m^+(\tilde{u}) = u^+$.

Proof of i): First, let us remark that it is well-known that I_+ is an open set (see for example D. Henry [11]). To prove i), we are going to exhibit, for each u_0 in X, an open set O_{u_0} such that:

i) $O_{u_0} \subset I_+ \cup I_0$, ii) $u_0 \in \bar{O}_{u_0}$.

Indeed if $u_0 \in I_+$, we take $O_{u_0} = I_+$. Next, if $u_0 \in I_-$, we have just seen that there exists $\varepsilon > 0$ such that:

$$\forall v \in X, \ u_0 \leq v \leq u_0 + \varepsilon\delta \Longrightarrow \omega(v) = \{u^+\}(= \{m^+(w)\}, \ \forall w \in \omega(u_0)) \ .$$

We then define:

$$O_{u_0} = \{v \in X, \ \exists 0 < \eta_1 < \eta_2 < \varepsilon, \ u_0 + \eta_1\delta \leq v \leq u_0 + \eta_2\delta \ \text{ in } \ \bar{\Omega}\} \ .$$

Obviously $u_0 \in \bar{O}_{u_0}$ and for all v in O_{u_0}: $\omega(v) = \{u^+\} \subset S_+ \cup S_0$ and thus $v \in I_+ \cup I_0$. Finally if $u_0 \in I_0$, two cases are possible;

1st case: $\exists v_n \underset{n}{\to} u_0,\ v_n > u_0,\ v_n \in I_-$. In this case we define:

$$O_{u_0} = \bigcup_n O_{v_n} \quad (O_{v_n} \text{ has been defined above}) .$$

Thus $O_{u_0} \subset I_+ \cup I_0$ and $u_0 = \lim\limits_n v_n \in \bar{O}_{u_0}$.

2nd case: $\exists \varepsilon_0 > 0,\ \forall v \in B(u_0,\varepsilon_0),\ v > u_0,\ v \not\equiv u_0 \Longrightarrow v \in I_+ \cup I_0$.

We then define

$$O_{u_0} = \{v \in X/\exists\, 0 < \eta_1 < \eta_2 < \varepsilon_0,\ u_0 + \eta_1\delta \le v \le u_0 + \eta_2\delta \text{ in } \bar{\Omega}\} \quad ;$$

and we conclude.

Remark II.6: i) Let $S_+ = \{u_j,\ 1 \le j \le N\}$ $(N \le +\infty)$, we may define:

$I_+^j = \{u_0 \in X/\omega(u_0) = \{u_j\}\}$. Then it is quite obvious to show that I_+^j are the connected components of I_+ (and thus are open)

ii) Let $S_- = \bigcup\limits_{1\le j\le M} C_j$ $(M \le +\infty)$ where C_j are the connected components of S_-, we may define:

$$I_-^j = \{u_0 \in X/\omega(u_0) \subset C_j\} \quad .$$

It is also easy to show that I_-^j are the connected components of I_- and that for all $u_0 \in I_-^j$, there exists $\varepsilon > 0$ such that:

$$\begin{cases} \forall v \in B(u_0,\varepsilon),\ v > u_0,\ v \not\equiv u_0 \Longrightarrow \omega(v) = \{u_j^+\} \\ \forall v \in B(u_0,\varepsilon),\ v < u_0,\ v \not\equiv u_0 \Longrightarrow \omega(v) = \{u_j^-\} \end{cases}$$

where u_j^+ is the minimum solution of (SP) above any element of C_j and u_j^- is the maximum solution of (SP) below any element of C_j (and $u_j^+, u_j^- \in S_+ \cup S_0$).

III. Variants and extensions:

III.1. Unbounded nonlinearities:

If we no more assume (9), the solution may not exist for all time. One way to get rid of this difficulty is to restrict our attention to an invariant domain K such that for any $u_0 \in K$, the solution of (IVP) exists for all $t > 0$ and remains bounded as $t \to \infty$. Then Theorem II.1 remains valid for u_0 in K (and for the relative topology of K in X).

Let us give two natural examples:

(i) $K = \{u_0 \in X/u(\cdot,t)$ exists for all $t > 0$, $\|u(\cdot,t)\|_{C(\bar\Omega)} \le C = C(u_0)$ $\forall\, t > 0\}$,

(remark that $K \ne \emptyset$ if and only if $S \ne \emptyset$);

(ii) $K = \{u_0 \in X/\underline{u} \le u_0 \le \bar u$ in $\bar\Omega\}$

where $\underline{u}$, $\bar u$ are sub and supersolutions of (SP) that is satisfying:

$$(18)\qquad \begin{cases} -\Delta\underline{u} \le f(\underline{u}) \text{ in } \Omega,\ -\Delta\bar u \ge f(\bar u) \text{ in } \Omega \\ \underline{u},\ \bar u \in C^2(\bar\Omega),\ \underline{u} \le \bar u \text{ in } \bar\Omega,\ \underline{u} \le 0 \le \bar u \text{ on } \partial\Omega \end{cases} .$$

In these two cases we define:

$$I_+ = \{u_0 \in K/\omega(u_0) \subset S_+\}$$
$$I_- = \{u_0 \in K/\omega(u_0) \subset S_-\}$$
$$I_0 = \{u_0 \in K/\omega(u_0) \subset S_0\} .$$

We then have:

Theorem III.1: *Let* $f \in C^2(\mathbb{R})$ *and let* K *be defined by (i) or (ii), we have:*

i) *There exists an open set* O *such that* $O \subset I_+ \cup I_0$, $\overline{O \cap K} = K$;

ii) *For all* u_0 *in* $I_+ \cup I_0$, $\omega(u_0)$ *is a singleton;*

iii) *If* $u_0 \in I_-$, *there exists* $\varepsilon > 0$ *such that:*

$$(13')\qquad \forall\, v \in B(u_0,\varepsilon) \cap K,\ v \ge u_0,\ v \not\equiv u_0 \Longrightarrow \omega(v) = \{u^+\} \subset S_+ \cup S_0$$

where $u^+ = m^+(w)$, $\forall\, w \in \omega(u_0)$;

$$(14')\qquad \forall\, v \in B(u_0,\varepsilon) \cap K,\ v \le u_0,\ v \not\equiv u_0 \Longrightarrow \omega(v) = \{u^-\} \subset S_+ \cup S_0$$

where $u^- = m^-(w)$, $\forall\, w \in \omega(u_0)$.

In particular if $v \in B(u_0,\varepsilon) \cap K$, $v \ge u_0$ *or* $v \le u_0$, $v \not\equiv u_0$ *then* $v \in I_+ \cup I_0$.

Remark III.1: The proof of this result is totally identical to the one of Theorem III.1. We just need to remark that if K is given by (i) and if there exist $u_0 \in I_-$, $v \in K$ such that $v \ge u_0$, $v \not\equiv u_0$ then $M^+(w)$ is not empty on the connected component of S_- containing $\omega(u_0)$.

<u>Remark III.2</u>: In some sense, the above result contains the results of P. L. Lions [13] (except some geometrical descriptions heavily dependent on the convexity of the nonlinearity f assumed in [13]).

<u>Remark III.3</u>: Most of the remarks made in the preceding section are still valid here (with some obvious adaptations).

III.2 Iterative schemes:

To simplify, we will consider the case of a nonlinearity satisfying (9) and we will assume in addition that we have:

(19) $$\exists K_0 \geq 0,\ f(t) + K_0 t \text{ is nondecreasing for } t \geq 0 .$$

We consider the asymptotic behaviour of iterative schemes like: $u_0 \in X$ and $(u_n)_{n \geq 1}$ is defined by

(20) $$-\Delta u_n + K u_n = f(u_{n-1}) + K u_{n-1} \text{ in } \Omega,\ u_n \in C^2(\bar{\Omega}),\ u_n = 0 \text{ on } \partial\Omega .$$

From (9), it is easy to deduce that u_n is bounded in $L^\infty(\Omega)$ and thus in $C^{2,\alpha}(\bar{\Omega})$ $(0 < \alpha < 1)$.

Now, if $K \geq K_0$ we have, multiplying (20) by $u_n - u_{n-1}$ and integrating by parts over Ω:

$$\int_\Omega |Du_n|^2 + K u_n^2 dx - \int_\Omega Du_n \cdot Du_{n-1} + K u_n u_{n-1} dx =$$
$$= \int_\Omega (f(u_{n-1}) + K u_{n-1})(u_n - u_{n-1})$$

and because of (19), this is less than:

$$\leq \int_\Omega F(u_n) + \frac{K}{2} u_n^2\, dx - \int_\Omega F(u_{n-1}) + \frac{K}{2} u_{n-1}^2\, dx$$

(recall that $F(t) = \int_0^t f(s)ds$). On the other hand, since we have:

$$\int_\Omega Du_n \cdot Du_{n-1} + K u_n u_{n-1} dx = \frac{1}{2}\int_\Omega |Du_n|^2 + K u_n^2 dx + \frac{1}{2}\int_\Omega |Du_{n-1}|^2 + K u_{n-1}^2 dx$$
$$- \frac{1}{2}\int_\Omega |D(u_n - u_{n-1})|^2 + K(u_n - u_{n-1})^2 dx \quad ;$$

we finally obtain, if $K \geq K_0$,:

$$\int_\Omega \frac{1}{2} |Du_n|^2 - F(u_n)dx - \int_\Omega \frac{1}{2} |Du_{n-1}|^2 - F(u_{n-1})dx \le$$

$$\le -\frac{1}{2} \int_\Omega |D(u_n - u_{n-1})|^2 + K(u_n - u_{n-1})^2 dx$$

and this enables us to show that: $u_n - u_{n-1} \underset{n}{\to} 0$ in $C^{2,\alpha}(\bar{\Omega})(0 < \alpha < 1)$. Therefore, if we denote by $\omega(u_0)$ the set of limit points of the sequence $(u_n)_{n\geq 1}$, we have:

(21) $\omega(u_0)$ is a compact, connected set, contained in S .

Therefore we may define again:

$$I_+ = \{u_0/\omega(u_0) \subset S_+\},\ I_- = \{u_0/\omega(u_0) \subset S_-\},\ I_0 = \{u_0/\omega(u_0) \subset S_0\} .$$

Of course I_+, I_-, I_0 are disjoint and $I_+ \cup I_- \cup I_0 = X$.

Then, we have:

<u>Theorem III.2</u>: <u>Under assumptions</u> (9), (19) <u>and if</u> $K > K_0$, <u>then we have</u>:

i) $I_+ \cup I_0$ <u>contains an open dense set,</u>

ii) <u>For all</u> u_0 <u>in</u> $I_+ \cup I_0$, $\omega(u_0)$ <u>is a singleton,</u>

iii) <u>If</u> $u_0 \in I_-$, <u>there exists</u> $\varepsilon > 0$ <u>such that</u>:

(13) $\forall v \in B(u_0,\varepsilon),\ v \geq u_0,\ v \not\equiv u_0 \Longrightarrow \omega(v) = \{u^+\} \subset S_+ \cup S_0$

<u>where</u> $u^+ = m^+(w)$, $\forall w \in \omega(u_0)$;

(14) $\forall v \in B(u_0,\varepsilon),\ v \leq u_0,\ v \not\equiv u_0 \Longrightarrow \omega(v) = \{u^-\} \subset S_+ \cup S_0$

<u>where</u> $u^- = m^-(w)$, $\forall w \in \omega(u_0)$.

<u>In particular if</u> $v \in B(u_0,\varepsilon)$, $v \geq u_0$ <u>or</u> $v \leq u_0$, $v \not\equiv u_0$ <u>then</u> $v \in I_+ \cup I_0$.

The proof of this result is identical to the one of Theorem II. 1 and we will skip it. In addition, all remarks made in the preceding sections are still valid with some obvious modifications.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2306	2. GOVT ACCESSION NO. AD-A114 624	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Structure of the Set of Steady-State Solutions and Asymptotic Behaviour of Semilinear Heat Equations		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) P. L. Lions		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE December 1981
		13. NUMBER OF PAGES 22
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

semilinear heat equations, semilinear elliptic equations, steady-state solutions, maximum principle

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We give a precise geometrical description of the set of steady-state solutions for general classes of semilinear heat equations. This enables us to prove global results about the asymptotic behaviour of the solutions of the initial value problem.

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